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Quantum Sets and Clifford Algebras

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The mathematical language presently used for quantum physics is a high-level language. As a lowest-level or basic language I construct a quantum set theory in three stages: (1) Classical set theory, formulated as a Clifford algebra of "S numbers" generated by a single monadic operation, "bracing," $\mathrm{Br} = \{\cdots\}$. (2) Indefinite set theory, a modification of set theory dealing with the modal logical concept of possibility. (3) Quantum set theory. The quantum set is constructed from the null set by the familiar quantum techniques of tensor product and antisymmetrization. There are both a Clifford and a Grassmann algebra with sets as basis elements. Rank and cardinality operators are analogous to Schroedinger coordinates of the theory, in that they are multiplication or "Q-type" operators. "P-type" operators analogous to Schroedinger momenta, in that they transform the Q-type quantities, are bracing (Br), Clifford multiplication by a set X, and the creator of X, represented by Grassmann multiplication $c(X)$ by the set X. Br and its adjoint Br^* form a Bose–Einstein canonical pair, and $c(X)$ and its adjoint $c(X)^*$ form a Fermi–Dirac or anticanonical pair. Many coefficient number systems can be employed in this quantization. I use the integers for a discrete quantum theory, with the usual complex quantum theory as limit. Quantum set theory may be applied to a quantum time space and a quantum automaton.

1. INTRODUCTION

Several of us here, including Feynman, Fredkin, Kantor, Moussouris, Petri, Wheeler, and Zuse, suggest that the universe may be discrete rather than continuous, and more like a digital than an analog computer. C. F. von Weizsaecker has worked this path since the early 1950s, and recently I have benefitted from the relevant work of J. Ford.

Von Neumann points out that quantum theory revises the predicate algebra of physics, making it coherent (in the sense of Jauch that it admits

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quantum superposition, is nondistributive). Therefore we must at the outset of theorizing choose the logic of our theory. A logic with superposition permits a kind of synthesis of the discrete and the continuous; it is well known. Therefore it is natural both physically and mathematically to seek a quantum mathematics constructed over the quantum predicate algebra in whatever sense classical mathematics is constructed over the Boolean predicate algebra. In particular, set theory has become the nearly universal language for mathematics, computer theory, and mathematical physics. Therefore it is appropriate first to make a quantum set theory.

Informal steps in that direction were made in "Space-Time Code" and subsequent papers. These overlooked the fact, adduced by Aristotle and developed by Kripke and others, that a language for a science, as opposed to a language for data recording, needs a concept of the possible. To express possibility the present quantum set theory uses the concept of indefinite object and indefinite set, and was stimulated by work of Solovay and Scott on random sets and Takeuti on quantum sets. With these concepts the construction becomes straightforward, and is sketched here.

In some earlier work we try to transpose field theory from classical to quantum time space, for a unitary quantum field theory. This transposition is awkward in that it requires a concept of exponential of quantum sets not available until recently, and is based on a serious misconception. I am now sure that "unitary field theory" is a contradiction in terms. A unitary theory cannot be a field theory. In a field theory the local topology of time space is almost independent of the contents of time space (being by hypothesis a Minkowskian manifold of four dimensions, say). Thus field theories dualistically divide (time space) form from content.

A unitary theory might have as its subject only the topological pattern of causal relations. This causal pattern, statistically described by the antimetric field of gravity, would be maximally described by discrete topological structure in such a unitary quantum theory. Field theory is adequate as a phenomenological description of experience where the topology and measure of time space do not participate in the physical process. A dynamics of the smallest times should involve the topology of time space much as general relativity involves the antimetric of time space. Such quantum topological effects should dominate during the least times in the universe, as during the creation of some particles and of the infant universe. The conception of a purely topological quantum physics is vividly expressed by Bohm, by Penrose, by Misner, Thorne, and Wheeler, and by others.

Quantum set theory provides a formalism for a no-field theory, for a quantum topology. Its Fock space S is generated not by a time space field of creation operators but by one creation operator Br. Yet S has sufficient structure to describe both time space and its contents. Just as set theory creates from the null set a universe of discourse rich enough for modern mathematics, quantum set theory provides for the creation of the physical universe from one point.

The main lesson among the many I draw from the quantum set theory of Takeuti is that there are two steps, not one, from classical set theory to quantum. The right route is as follows: (1) classical sets: Boolean and definite; (2) semiclassical sets: Boolean and indefinite; (3) quantum sets: non-Boolean and indefinite.

The central idea of indefinite set theory is possibility, a term already applied in a like connection by Kripke and Bub. Each *object* is supposed to be associated with and identified with a set of possibilities called its *scope*. An indefinite object proper is one whose scope contains more than one possibility. A definite object has only one possibility.

To emphasize the involvement of the whole system in the subsystem studied, I call quantum possibilities *choices*, distinguishing between initial choices, represented by choice vectors or kets, in the terminology of Dirac; and final choices, represented by choice covectors (bras). Quantum superposition applies only to choices (of the same tense), not to objects.

When we deal with quantum systems, we replace the scope of the classical indefinite objects, with its commutative function algebra and its distributive subset lattice, by a linear space, or slightly more generally by a module over a ring, with a noncommutative operator algebra and a nondistributive subspace lattice.

An indefinite set is an indefinite object whose possibilities are sets.

The main mathematical difference between indefinite set theory and other set theories such as fuzzy (Zadeh), hazy (Dodson), random (Solovay), and those of Takeuti, is how the hierarchy of sets is generated. Indefinite set theory uses the tensor product to generate the algebra of projections inductively, concurrently with the generation of the sets, while fuzzy, random, and Takeuti quantum sets use a quite arbitrary fixed algebra of projections, chosen at the start.

I make three simplifying restrictions in this presentation. I consider only those sets that are made from the null set by the operations of set theory. I employ only the integers

$$\mathrm{Int}=\{\ldots,-2,-1,0,1,2,\ldots\}$$

for coefficient numbers or amplitudes, building an *integer quantum theory* more restricted in its choices than complex quantum theory, the theory with the usual complex amplitudes. I consider only finite sets.

Such a theory (null foundation, integer coefficients, finite sets) is best for a basic physical theory, a low-level language, and the more elaborate theories (nonnull foundation, infinite sets, coefficients in an algebra) are better for more phenomenological theories, higher-level languages. In principle any ring with star ($*$) and quasi-inverses (any regular *algebra) may be a candidate for the c numbers of quantum physics, and some hypercomplex numbers are of physical interest in gauge theories.

2. DEFINITE SET THEORY

In this section we summarize the naming of the sets, the operation of set multiplication, and the set functions of rank and cardinality. I say some things three times: in standard brace-and-comma formulas, in "box algebra," and in a Clifford algebra of S numbers.

2.1. Brace-and-comma Formulas for Sets. We generate Set, the set of all finite sets, recursively by means of a single polyadic operation $\{s, s', \ldots\}$. (An n-adic function has n arguments and a polyadic function has a sequence of any number of arguments.) Here we apply this operation only to a finite sequence of arguments and only a finite number of times.

Suppose $s', s'', \ldots$ are sets that have been generated at stage n of the recursion. Then the sets generated at the next stage include these and also the set s given by A1:

A1. $s := \{s', s'', \ldots\}$. We identify the formulas resulting from A1 in equivalence classes generated by A2:

A2. *Commutativity*: A permutation of the arguments $s', s'', \ldots$ in A1 produces an equivalent result. *Absorption*: Replacing any subsequence $s', s', \ldots$ of repeated arguments by one s' produces an equivalent result.

For example, $\{A, B, A\} = \{A, B\}$, where two A's have been absorbed into one.

I write this recursion as A1/A2. Such a symbolic quotient means that the numerator generates formulas and the denominator identifies formulas.

When we begin generating sets, there are no "already generated" sets, and the sequence of sets in A1 must be null. A1 must still be applied, and generates the set $\{\ \}$, the *null set*.

When

$$s = \{a, b, \ldots\} \tag{1}$$

we say that a is a member of s, $a \epsilon s$, a is in s, and s' contains a. If $a \epsilon a' \epsilon \ldots a''$, we say that a is embraced by a'', writing $a \epsilon^* a''$. No set embraces itself.

The recursion A1 generates formulas consisting entirely of left braces, commas, and right braces, called "brace-and-comma formulas" here. Any physical or mathematical process having the properties A2 of $\{\ldots\}$ is a possible interpretation for set theory.

The most common interpretation of (1), which may be considered the standard interpretation, is that s is the "concept" of $a, b, \ldots$. Insofar as concepts are creations of minds, this is a mentalist interpretation; insofar as sets are supposed to exist as ideas, an idealist interpretation. Alternatively, a set may be interpreted as an equivalence class of synonymous formulas; the name of its members; a list; or a possible physical assembly.

2.2. Sets as Causal Spaces. We also have in mind a somewhat novel interpretation of (1). Sets $a, b, \ldots$ may be interpreted as basic events and s as their (immediate causal) *consequent*. The importance of causal order as a possible primitive concept has been emphasized by Robb, Alexandrov, Penrose, and others. In earlier studies we represented the causal relation by set inclusion acs, not by membership $a \epsilon s$. The greater structure of the present theory is supposed to replace the fields that had to be added in the earlier ones.

Each set with the sets it embraces is then a possible pattern of events or causal space and defines a graph, a subgraph of the graph of ϵ.

Such a universe has an initial event, the null set; a final event, the set itself; and no time loops (closed timelike paths). While these three properties of sets as universes are physically plausible, many universes obeying Einstein's law of gravity lack these properties.

We could describe time loops if, as in the early days, sets were allowed reciprocal membership (as once one said that $1 \epsilon \{1\}$ and $\{1\} \epsilon 1$; that the "set of unit sets" 1 is a member of and also contains the set whose only member is 1) and self-membership (as once one said that the universal "set" contained itself $V \epsilon V$).

2.3. Box Algebra of Sets. The set construction A1 is conveniently factored into two, more elementary, processes, dyadic "boxing" $a \Box b$, and monadic "bracing" $\{a\}$.

$a \Box b$ is a group product and coincides with the Boolean sum (= symmetric difference = exclusive or). The Boolean "sum" *must* be recognized as a product. A true sum $a + b$ is introduced later, and the box product $a \Box b$ distributes over that sum.

$\{a\}$ is the unit set or singleton containing a. We also write $\{a\}$ with an operator Br defined by

$$\mathrm{Br}(a) := \{a\} \tag{2}$$

We define

$$\mathrm{Br}'(s) := \{\mathrm{Br}(x) \mid x \epsilon s\} \tag{3}$$

Let us now think of Set as an algebra with two operations, Br and $\Box$, to be generated by the following recursion B1/B2.

An important point of notation: Since the formula "{ }" is the identity for the box product $a\Box b$, I designate { } by 1.

The semicolon in B1 means a disjunction of possible recursions.

B1. $s{:}=1;\ \{s'\};\ s'\Box s''$.

B2. $s\Box s'=s'\Box s,\quad s\Box(s'\Box s'')=(s\Box s')\Box s'',\quad 1\Box s=s,\ s\Box s=1$.

The interpretation of the box and brace processes is determined by the interpretation of the set theory. In the causal interpretation of set theory, the brace Br increases proper time by one elementary unit of time; one τ, I say for short. Presumably,

$$10^{-43}\ \text{sec} \lesssim 1\ \tau \lesssim 10^{-23}\ \text{sec} \tag{4}$$

Evidently Br will be closely related to energy, the time generator, in the causal interpretation. Schematically speaking, we expect

$$\text{Br}=\exp(\text{i}E)$$

in units where τ and $\hbar$ are 1, with i defined by this too.

$a\Box b$, in the causal interpretation, is the consequence of the inverse consequences of a and b. This rarely exists in familiar time spaces.

2.4. Clifford Algebra of S Numbers. For brevity I call a free additive group (a "linear space over the integers") a module. By an inner product $m\cdot m'$ on a module M, I mean a symmetric bilinear function on $M\times M$ to Int.

By a submodule of the module M we shall mean a subset N of M that is closed with respect to linear dependence as well as linear combination. Example: The module of even integers 2Int is *not* a submodule of the module of integers Int. Proof: 2Int does not contain 1, yet 1 is dependent on 2Int: $2\times 1=2$.

The Clifford algebra $C=C(M)$ of a module M with inner product $m\cdot m'$ is defined by the following recursion C1/C2:

C1. $c{:}=1;\ m;\ c'c'';\ c'+c'';\ -c'$.

C2. $c(c'c'')=(cc')c''.\ \ c1=c.\ \ (c+c')+c''=c+(c'+c'').\ \ c+0=c.\ \ -c+c=0.\ c+c'=c'+c.\ mm'+m'm=2m\cdot m'.\ c(c'+c'')=cc'+cc''.$

There is a well-known expression of $C(M)$ as a sum of submodules $C=C(0)+C(1)+C(2)+\cdots$ with $C(0)=\text{Int}$ and $C(1)=M$. The function $C(n)$ assigning a submodule to each number $n=0,1,\ldots$ is called the *grade* of the Clifford algebra. $C(n)$ is uniquely defined as the submodule spanned by C numbers expressible as products of n M numbers but not $n-1$ M numbers. C numbers in $C(n)$ are said to have grade n.

The module M also defines a Grassmann algebra $G(M)$, with the same members as $C(M)$, by a recursion G1/G2 that differs from C1/C2 only in that the products xy are all replaced by the wedge product $x \wedge y$, the inner product $m \cdot n$ is replaced by 0, and the relation

$$mm' = m \wedge m' + m \cdot m'$$

is imposed between Clifford product mm' and Grassmann product $m \wedge m'$ of M numbers.

The spectacular physical application of Clifford algebra is the Dirac equation. The importance of Clifford algebra for basic physics has been emphasized by Eddington ("E numbers"), Riesz, Hestenes, and Dresden, and the mathematical theory has been developed by Chevalley.

We define the important Clifford algebra S, later representing the quantum set; its submodule $S(1)$, later representing the quantum unit set (cardinality 1); a preferred redundant S basis B $(= -B)$, representing the classical or Boolean set; a preferred redundant $S(1)$ basis $B(1)$, representing the Boolean unit set; and the integers Int, an S ray also written Int $= S(0)$; with generic members s in S, b in B, r in $S(1)$, a in $B(1)$, and i in Int, by the following recursion $S1/S2$, a specialization of C1/C2:

S1. $b:=1: \{b'\};\ b'b'';\ -b'.$
$s:=b'+b''.$
$a:=\{b'\}.$
$r:=a'+a''.$
$i:=1;\ i+i';\ -i'.$

S2. $s(s's'')=(ss')s''.\ \ si=is=s.\ \ s+(s'+s'')=(s+s')+s''.$
$s+s'=s'+s.\ \ s+0=s.\ \ -s+s=0.\ \ s(s'+s'')=ss'+ss''.$
$\{s+s'\}=\{s\}+\{s'\}.\ \ \{-s\}=-\{s\}.$
$rr'+r'r=2r\cdot r'.$
$a\cdot a=-1.$
$a\cdot a'=0$ unless $a'=a$ or $-a$.

The integers Int are S numbers and must be separated from the integers of von Neumann, which are sets of increasing rank and cardinality.

We interpret S numbers recursively. Let S numbers $s, s', \ldots$ represent sets $x, x', \ldots$. Then

I1. 1 represents the null set.

I2. $\{s\}$ represents $\{x\}$.

I3. $ss'\ldots$ represents the Boolean sum $x \square x' \ldots$.

I4. is $(i \neq 0)$ represents x.

S numbers not interpreted by I1–4 are meaningless until Section 4. Thus 0 names no set.

$S = C(S(1))$. $S(1)$ is the module underlying S.

In the following paragraphs we adduce some operations on definite sets, for generalization later to indefinite sets.

2.5. Grade and Cardinality. Let $S(n)$ be the grade of S.

Grade on S expresses the set theory concept of cardinality. If S number s represents a set x then the grade of s is $\text{card}(x)$, the number of members of x. We write $\text{card}(s)=n$ for nonzero S numbers in $S(n)$.

B numbers of cardinality $0,3,4,7,8,\ldots$ have square $+1$; the rest, -1.

2.6. Inner Product. The inner product $r\cdot r'$ of two $S(1)$ numbers r, r' is defined by $S1/S2$. Any set is represented by two B numbers, and distinct sets by orthogonal B numbers. The grade assigns to each S number s an integer $\text{Av}(s)$ defined as the projection of s onto $S(0)$. That is, $\text{Av}(a\{\ \}+b\{\{\ \}\}+\cdots)=a$ for $a,b,\ldots$ in Int. I define an indefinite bilinear form $s\cdot s'$ on S by

$$s\cdot s'=\text{Av}(ss')$$

This form extends the form $m\cdot m'$ already defined for $S(1)$ numbers. Evidently $\text{Av}(X)=X\cdot 1$. There also exists a unique positive definite inner product s^*s' on S such that

$$b^*b=1, \qquad b^*b'=0$$

for all B numbers b, b' with $b'\neq +b,-b$.

2.7. Creator–Destructor of a Set. If s is any S number, s defines a linear operator on S by left multiplication, mapping any S number t into st. Let s, t, and st represent sets x, y, and z, respectively. If y includes x then z does not, and conversely. Thus s may be called a creator–destructor of x. In the Grassmann case ($a\cdot a=0$ instead of -1 in $S2$) s creates x or annihilates the operand entirely, and is called the x creator.

2.8. Adjoint. We use the metric s^*s' on S to define an *adjoint* L^* for any endomorphism ("integer linear operator") L (when L^* exists) by requiring for all S numbers s and s' that

$$s'\cdot L^*s=s\cdot Ls'$$

We speak, in obvious analogy to the usual Hilbert space terminology, of Hermitian and anti-Hermitian operators, normal operators, isometric operators, and unitary operators on S.

2.9. Rank of a Set. The *rank* of a B number b is a natural number we will write as $\text{rank}(b)$. Rank is the maximum number of nested braces. We define rank recursively for B thus: Let $b, b', \ldots$ have ranks $n, n', \ldots$. Then

R1. $\text{rank}(1)=1$.

R2. $\text{rank}(\{b\})=n+1$.

R3. If $\text{card}(bb')=\text{card}(b)+\text{card}(b')$ then $\text{rank}(bb')=\sup(n, n')$.

In the causal interpretation, the rank of a set means the proper time interval between the first and last events in the universe represented by the set, measured in π of equation (4). This identification depends upon and incorporates the mathematically curious feature of relativity that timelike geodesics are curves of greatest, not least, interval.

3. INDEFINITE SET THEORY

One example of a Boolean indefinite set is an indefinite object ISet whose scope is Set.

We may define for indefinite sets the relations and operations already defined for definite sets, the new concept of multiplicity, and the new operation of addition. In this classical statistical theory we make use of the fact that indefinite sets are themselves merely indefinitely described definite sets, as a kind of scaffolding for the construction. In the following section, on quantum sets, we make the quantum leap from the scaffolding, into superposition.

We designate by $\text{sc}(A)$ the scope of object A. Conversely, we write $\text{ob}(S)$ for the indefinite object whose scope is S. Thus ISet=ob(Set). In anticipation of quantum practice, we represent all scopes as modules ("linear spaces over the integers") provided in the case of Boolean objects with preferred bases consisting of the possibilities for the indefinite objects. A module with a preferred basis we call a "based module" in what follows.

Generally speaking, we define an operation on indefinite objects by performing the operation on the possibilities of the objects and assembling the results into a scope. Some of the following definitions illustrate this process.

3.1. Multiplicity of an Indefinite Object. Indefinite objects have a parameter that is identically 1 for definite objects, *multiplicity*. In spectroscopy the quantum analog of multiplicity is also called degree of degeneracy, and "nondegenerate" means "having multiplicity 1." Multiplicity is the dimension of the scope:

$$\text{mult}(B)=\dim(\text{sc}(B))$$

Sets may now have three integer parameters of interest, rank, cardinality, and multiplicity. They are independent parameters, and while multiplicity is always defined, rank, and cardinality are not.

We henceforth identify definite objects with indefinite objects of multiplicity 1.

Example: A pair of dice may be described as an indefinite object of rank 1 (being a set of nonsets), cardinality 2 (being a pair), and multiplicity 36 (there being 36 possibilities). A coin, likewise, has rank 0 and cardinality 0 (having no members) and multiplicity 2. On the other hand, an indefinite object that is either a coin or a pair of dice has indefinite rank, indefinite cardinality, and multiplicity 38. The scopes of these indefinite objects are based modules of dimensions 36, 2, and 38 (over the integers).

3.2. Sum of Indefinite Objects. For any indefinite objects (including indefinite sets) $A, B, \ldots$ we define the sum $A+B$ by giving the based module $\mathrm{sc}(A+B+\cdots)$ as a (direct) sum of the based modules $\mathrm{sc}(A)$, $\mathrm{sc}(B), \ldots$:

$$\mathrm{sc}(A+B+\cdots) := \mathrm{sc}(A)+\mathrm{sc}(B)+\cdots \tag{5}$$

In this equation we add modules by forming sums of the respective module members in all possible ways. We must still, however, give a basis for the sum. For objects with scopes disjoint except for 0, we define the basis of the sum to be the union of the bases of the terms. This inelegance disappears in the quantum theory.

The sum of indefinite objects is commutative and associative and has the identity object 0 whose module consists of the number 0.

This enables us to express any indefinite object as a sum of indefinite objects of multiplicity 1, or definite objects.

For disjoint terms, the multiplicity of a sum is the sum of the multiplicities of the terms:

$$\mathrm{mult}(A+B+\cdots) = \mathrm{mult}(A)+\mathrm{mult}(B)+\cdots$$

It is simple now to define the product and exponential of indefinite sets. Using the sum and product we may recursively define a class of indefinite sets in close correspondence to the recursion $S1/S2$ given for definite sets. I omit details for reasons of space.

The product of indefinite objects is commutative and associative and distributes over the already defined sum of indefinite objects. The object ob(1) whose module is Int is the identity for this product.

3.3. Brace of an Indefinite Set. We define $\{A\}$ for an indefinite set A by giving the scope (based module) of $\{A\}$. We take

$$\mathrm{sc}(\{A\}) := \mathrm{Br}'(\mathrm{sc}(A))$$

We define the sum and difference of $\{a\}$ and $\{a'\}$, for any members a, a' of $\mathrm{sc}(A)$, by

$$\{a\}+\{a'\}=\{a+a'\}, \qquad \{-a\}=-\{a\}$$

so that bracing is an endomorphism. We define the basis of the based module $\mathrm{sc}(\{A\})$ by bracing the members of the basis of $\mathrm{sc}(A)$. Bracing does not change multiplicity: $\mathrm{mult}(\{A\})=\mathrm{mult}(A)$.

4. QUANTUM SET THEORY

I call possibilities that admit superposition "choices."

A quantum object is an indefinite object with choices. In the most common language, a quantum object is represented by a scope that is a linear space or, slightly generalized, a module. There is much to say for the position, especially developed by Segal, that the best space to represent quantum objects is an algebra, but for maximum familiarity I use linear spaces here. For economy I consider only the quantum object QSet whose scope is the module S defined by the recursion $S1/S2$. No preferred basis is assumed for quantum objects in general.

The usual way to make the linear space and algebra of coordinates of a quantum theory, given a classical theory, is a device used by Frobenius. If a group acts as a set of transformations (typically nonlinear) on a set X (typically the scope of some system), then Frobenius constructs a (linear) representation of the nonlinear group by forming a space $L(X)$ of functions on X and (in the simplest instance) letting the group act in the natural way on this function space. If $f(x)$ is a function in $L(X)$, then a group element g maps $f(x)$ into $f'(x)$ defined by

$$f'(x')=f(x) \qquad \text{for } x'=g(x) \tag{6}$$

We regard S as the result of applying the Frobenius construction to $X=B$, whose members represent Set doubly.

We now have an interpretation for the operation of addition in S previously uninterpreted. We add to the rules of interpretation I1–I4 the rule

I4 cont. And $u+u'$ represents a quantum superposition of the choices s and s'.

Physical operators of any quantum theory are built out of basic operators of two kinds, which I will call Q-type and P-type operators. A Q-type operator on $L(X)$ is a multiplication by a function $q(x)$ in $L(X)$. Such a function usually represents a physical quantity q in the classical theory with the scope X. A P-type operator on $L(X)$ is a linear transformation of $L(X)$ defined by a map of X into X, not a real function on X, in the manner of equation (6). In a classical theory using the scope X, a map of X into X does not represent a physical quantity, but there is often a natural way to construct such a map from a physical quantity; for example, via symplectic structure.

It is important and not always easy to choose the right scope X before applying the Frobenius construction. To get quantum mechanics from classical mechanics, we take X to be the configuration space or q space, not the phase space or (p, q) space, and take the coordinate operators to be Q-type, but the momentum operators to be P-type, associated with infinitesimal translations of q space.

Each S number is now supposed to represent a possible choice for the quantum object QSet $=$ ob(S). Each Q-type operator retains the interpretation of the function on S. For example, the functions rank and card on Set define corresponding functions on B which in turn give rise to Q-type operators on S of multiplication by the functions rank(b) and card(b), if we represent each S number c by the amplitude $a(b)=b\cdot c$. The operators rank and card are interpreted in the usual quantum way as the rank and cardinality of the (indefinite) quantum set.

On the other hand, the brace $\mathrm{Br}(b)=\{b\}$ gives rise to a P-type quantity represented by the endomorphism (integer linear operator) B defined by Br $c=\mathrm{Br}(c)=\{c\}$. Formally, the quantum operator Br braces the entire S number c on which it acts, and is reduced by linearity to a sum of braces of basic members.

The interpretation of P-type quantities is based on their algebraic properties, especially their commutation relations with Q-type quantities.

For example rank is a Q-type quantity. The brace creator Br is also a rank creator in that

$$(\mathrm{rank})\mathrm{Br}=\mathrm{Br}(\mathrm{rank}+1)$$

(This tells us that Br maps an eigennumber of the operator rank into another eigennumber of rank with eigenvalue greater by 1.) To construct observables we require normal operators.

We compute for this purpose the adjoint of Br. Since Br increases rank by 1 and results in a set of cardinality 1, Br* annihilates all basis members of cardinality other than one (nonunit sets), and maps any unit set into its member. Thus Br* is a rank destructor and a brace destructor.

Let us write [unit] for the projection operator or *projector* on $S(1)$ and 1-[unit]=[nonunit] for the projector on nonunit sets. It is evident that

$$\mathrm{Br}^*\mathrm{Br}=1$$

$$\mathrm{Br}\mathrm{Br}^*=[\mathrm{unit}]$$

$$\mathrm{Br}^*\mathrm{Br}-\mathrm{Br}\mathrm{Br}^*=[\mathrm{nonunit}] \tag{7}$$

The usual Bose–Einstein relations between the creator and destructor of braces, or more accurately of rank, is Br*Br − BrBr* = rank (not valid here). The normalization factor that would be required to convert our Br into an operator with this more familiar property involves a square root that is not available within the integers, but will be available at a later stage, when the reals have been introduced. We may take equation (7) as the integer form of Bose–Einstein relation.

We see that Br, being nonnormal, cannot itself represent an observable quantity according to the usual principles of quantum theory. Br gives rise to such normal operators as Br + Br*, Br − Br*, and Br*Br. The operator Br is an isometry.

A B number b is not a creator of b, and we have called b a creator–destructor of b. For a b creator in the usual sense we define the operator of left *Grassmann* multiplication with b, designating this operator by $c(b)$:

$$c(b)m=b\wedge m$$

Let $n(b)$ be the (Q-type) projector on the submodule generated by sets containing b. [$n(b)$ has eigenvalue 0 on sets lacking b and 1 on sets containing b.] It is easy to see that $c(b)^*$ annihilates any basis member that does not contain b, and that

$$c(b)c(b)^*=n(b)$$

$$c(b)^*c(b)=1-n(b)$$

$$c(b)c(b)^*+c(b)^*c(b)=1$$

the usual Fermi–Dirac relation between creator and destructor. In the Fermi–Dirac case, there is no problem with square roots. Fermi–Dirac relations are more natural for integer coefficients than Bose–Einstein.

There is an algebra of quantum sets much like the algebra of indefinite sets.

5. DISCUSSION

We have formally quantized set theory. The result is an algebra (in the general sense) based on three operations (subtractive) superposition $X - Y$, brace $\mathrm{Br}(X)$, and Clifford product XY. $\mathrm{Br}(X)$ and XY come from classical set theory, and $X - Y$ from quantization.

As an exercise in applying this algebra we interpret sets as events and membership as immediate causal relationship in a quantum topology.

Indeed if the theory of the causal relation of the universe were a quantum theory, this would account for the quantum coherence of nature as simply as possible.

The universe is relativistic as well. Without reference to symmetry groups, which must all be approximate, we can express the idea of relativity by saying that all assertions about time space must be expressed solely in terms of the pattern of causal relations. We have suggested a scope S for the universe with definite dimension and inner product in order to have a theory where we could formulate questions about the quite distinct dimension and (antimetric) inner product of the universe itself. These have more to do with one choice than with the whole scope and are next on my agenda.

Some difficulties of physics already appear. The universe of this theory, unlike Einstein's, must have an original event (the null set) and no time loops, lacks local Lorentz invariance, and may have huge dimension or none. I suspect the arrow on these graphs may be a vestige of macroscopic thermodynamics, and am exploring its elimination with a symmetric causal relation. Another plausible simplification replaces the two operations $\mathrm{Br}(X)$ and XY by one dyadic operation $(XY) = \mathrm{Br}(XY)$, a nonassociative product. This gives each event two antecedents as in the Feynman checkerboard and the "space-time code."

If we think of the pattern as the graph of a computer, relativity deals harshly with its hardware. Man-made computers are assemblies of persistent things with persistent interconnections. They thus define a rest system. It would mean giving up the principle of relativity of motion to identify the world with such a computer. The elements of the world computer are transient, and may be identified with world points or events themselves. If we persist in thinking of the world as a computer, we should imagine that each of its cells lasts but a τ.

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